Quantum renormalization group

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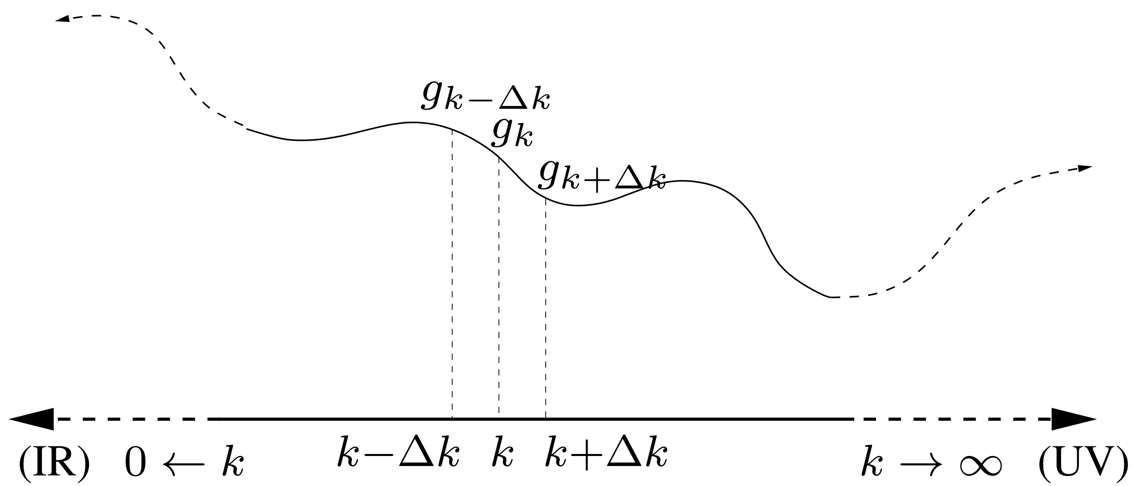
Renormalization group

- The functional renormalization group (RG) method is non-perturbative method in quantum field theory.
- The RG method creates a bridge between the UV and IR descriptions of the investigated model.

The evaluation of the path integral dresses up the values of the couplings with their corrections coming from the quantum fluctuations. The vacuum to vacuum transition amplitude is

$$Z = \int \mathcal{D}\phi e^{-S_k} = \int d\phi_0 \dots d\phi_{k-\Delta k} d\phi_k d\phi_{k+\Delta k} \dots d\phi_\infty e^{-S_k}$$

The path integral is performed by removing the degrees of freedom (quantum fluctuations, modes) one-by-one, systematically, which gives scale dependent couplings.



The Wegner-Houghton equation

The Wegner-Houghton equation is derived by the help of a new small parameter, $\Delta k/k$, where k is the cutoff which is lowered by Δk in a blocking step of the generating functional,

$$e^{W[j]} = \int D[\phi] e^{-S_E[\phi] + \int dx j(x)\phi(x)}.$$

The higher than one-loop contributions to the blocking relation for the Euclidean Wilsonian action $S_k[\phi]$ of the scalar field ϕ,

$$e^{-S_{k-\Delta k}[\phi]} = e^{-S_k[\phi+\chi] - \frac{1}{2}\mathrm{Tr}\ln\frac{\delta^2 S_k[\phi+\chi]}{\delta\phi\delta\phi} + \mathcal{O}((\Delta k/k)^2)},$$

where S_k is the blocked action and χ stands for the saddle point, the trace is taken in the momentum region $k - \Delta k < |p| < k$. We assume $\chi = 0$ in this work and calculate the trace in momentum space,

$$\mathrm{Tr}\ln\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi} = \int_{k-\Delta k<|p|<k} \ln\frac{\delta^2 S_k[\phi]}{\delta\phi_{-p}\delta\phi_p}.$$

The Wegner-Houghton equation

In the LPA one projects the equation into the functional space, defined by the ansatz

$$S_k[\phi] = \frac{1}{2}\int_x \phi_x D_0^{-1}\phi_x + \int_x U_k(\phi_x),$$

with $D_0^{-1} = p^2$. The Wegner-Houghton equation reads

$$\dot{U}_k = -\frac{k^3}{4\pi^2}\ln\left(k^2 + U_k''\right).$$

We further narrow the functional space where the solution of the evolution equation is sought by allowing the simple form,

$$U = \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4,$$

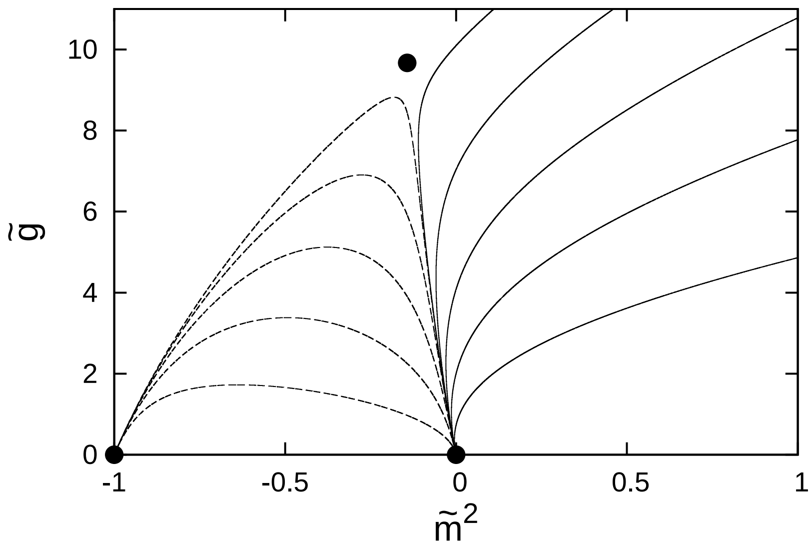
for the potential. Its evolution is given by the β functions

$$\begin{aligned}\dot{m}^2 &= \partial_\phi^2 \dot{U}_{k|\phi=0}\\ \dot{g} &= \partial_\phi^4 \dot{U}_{k|\phi=0}\end{aligned}$$

The 3d ϕ^4 model

Evolution equations

$$\dot{\tilde{m}}^2 = -2\tilde{m}^2 - \frac{\tilde{g}}{4\pi^2(1+\tilde{m}^2)},$$
$$\dot{\tilde{g}}^2 = -\tilde{g} + \frac{3\tilde{g}^2}{4\pi^2(1+\tilde{m}^2)^2}.$$



rescaling: $\omega = 1 + \tilde{m}^2$, $\chi = \tilde{g}\omega$ és $\partial_\tau = k\partial_k/\omega$

$$\partial_\tau \omega = 2\omega(1-\omega) - \frac{\chi\omega}{4\pi^2}$$
$$\partial_\tau \chi = -\chi(2-\omega) + \frac{\chi^2}{\pi}$$

UV: GFP, $\chi^*_G = 0,\ \omega^*_G = 1$
$\rightarrow\ \tilde{m^2_G}^* = 0,\ \tilde{g}^*_G = 0$

CO: WF, $\chi^*_{WF} = 8\pi^2/7,\ \omega^*_{WF} = 6/7$
$\rightarrow\ \tilde{m}^2_{WF} = -1/7\ \tilde{g}_{WF} = 48\pi^2/49$

IR: $\chi^*_{IR} = 0,\ \omega^*_{IR} = 0$
$\rightarrow\ \tilde{m^2_{IR}}^* = -1,\ \tilde{g}^*_{IR} = 0$

WH equation, Minkowski spacetime

We apply the cutoff for the space components of the momentum only and integrate over the frequency,

$$\mathrm{Tr}\ln\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi} = \int_\omega\int_{k-\Delta k<|\mathbf{p}|<k} \ln\frac{\delta^2 S_k[\phi]}{\delta\phi_{-\omega-\mathbf{p}}\delta\phi_{\omega\mathbf{p}}}.$$

The evolution of the potential is governed by the equation

$$\dot{U}_k = -\frac{1}{4\pi}k^2\int_\omega \ln\left(\omega^2+k^2+U_k''\right),$$

which reads as

$$\dot{U}_k = -\frac{1}{8\pi}k^2\sqrt{k^2+U_k''},$$

after the frequency integration, giving rise to slightly different β functions

$$\begin{aligned}\dot{\tilde{m}}^2 &= -2\tilde{m}^2 - \alpha_d\frac{\tilde{g}}{2(1+\tilde{m}^2)^{1/2}},\\ \dot{\tilde{g}}^2 &= (d-4)\tilde{g} + \alpha_d\frac{3\tilde{g}^2}{4(1+\tilde{m}^2)^{3/2}}.\end{aligned}$$

The flow equations provide the same qualitative picture.

Renormalization in CTP formalism

The CTP action has the form

$$S_k[\hat{\phi}] = \frac{1}{2}\int_x \hat{\phi}_x \hat{D}_0^{-1}\hat{\phi}_x - \int_x U_k(\hat{\phi}_x), \;\; \hat{D}_{0p}^{-1} = \begin{pmatrix} p^2 + i\epsilon & -2i\Theta(-p^0)\epsilon \\ -2i\Theta(p^0)\epsilon & -p^2 + i\epsilon \end{pmatrix},$$

The CTP WH equation can be get by replacing the inverse propagator in

$$\dot{U}_k = i\frac{k^2}{4\pi}\int_\omega \ln \hat{D}_{\omega,k}^{-1}.$$

The quartic potential ansatz is

$$\begin{aligned} U_k &= \frac{m^2}{2}\phi^{+2} - \frac{m^{2*}}{2}\phi^{-2} + i\mu^2\phi^+\phi^- + i\frac{h}{4}\phi^{+2}\phi^{-2} + \frac{\lambda}{6}\phi^{+3}\phi^- - \frac{\lambda^*}{6}\phi^+\phi^{-3} \\ &\quad + \frac{g}{4!}\phi^{+4} - \frac{g^*}{4!}\phi^{-4} \\ &= \frac{m^2}{2}\phi^{+2} - \frac{m^{2*}}{2}\phi^{-2} + i\mu^2\phi^+\phi^- + V(\hat{\phi}) \end{aligned}$$

Renormalization in CTP formalism

The evolution equations for the couplings are:

$$
\begin{aligned}
\dot{\tilde{m}}^2 &= -2\tilde{m}^2 + \Re\,\dot{\tilde{U}}^{++}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{\mu}}^2 &= -2\tilde{\mu}^2 + \Im\,\dot{\tilde{U}}^{+-}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{\lambda}}_r &= (d-4)\tilde{\lambda}_r + \Re\,\dot{\tilde{U}}^{+++-}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{\lambda}}_i &= (d-4)\tilde{\lambda}_i + \Im\,\dot{\tilde{U}}^{+++-}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{h}} &= (d-4)\tilde{h} + \Im\,\dot{\tilde{U}}^{++--}\Big|_{\phi_+=0,\phi_-=0} \\
\dot{\tilde{g}} &= (d-4)\tilde{g} + \Re\,\dot{\tilde{U}}^{++++}\Big|_{\phi_+=0,\phi_-=0} + 2\dot{\tilde{\lambda}}_r
\end{aligned}
$$

The phase structure remains the same as was obtained in the single time treatment with 2 phases

Tree level evolution

The bare theory is defined by the action,

$$S_B[\hat{\phi}] = \int dx \left[\frac{1}{2}\hat{\phi}_x \hat{D}_0^{-1}\hat{\phi}_x - \sum_{n=2}^{\infty}\frac{g_{Bn}}{n!}(\phi_x^{+n} - \phi_x^{-n})\right],$$

where we use the free massless propagator,

$$\hat{D}_0 = \begin{pmatrix} D_0^n + iD_0^i & -D_0^f + iD_0^i \\ D_0^f + iD_0^i & -D_0^n + iD_0^i \end{pmatrix},$$

given by $D_0^n = P(1/(p^2 - m^2))$, $D_0^f = -i\pi\delta(p^2 - m^2)\mathrm{sign}(p^0)$,
$D_0^i = -i(1 + 2n)\pi\delta(p^2 - m^2)$ in the Fourier space. The off-diagonal elements are on-shell.
The blocked action is

$$\begin{aligned} S[\hat{\phi}] &= S_c[\hat{\phi}] + S_o[\hat{\phi}] \\ S_c[\hat{\phi}] &= \int dx \left[\frac{1}{2}\hat{\phi}_x \hat{D}_0^{-1}\hat{\phi}_x - U(\phi_x^+) + U(\phi_x^-)\right], \quad U(\phi^\sigma) = \sum \frac{g_n}{n!}\phi^{\sigma n} \\ S_o[\hat{\phi}] &= -\int dxdy V_{x-y}(\hat{\phi}_x, \hat{\phi}_y), \quad V_{x-y}(\hat{\phi}_x, \hat{\phi}_y) = \sum_{\sigma,\sigma'}\sum_{mn\geq 3}\frac{1}{m!n!}\phi_x^{\sigma m} h^{\sigma,\sigma'}_{m,n,x-y}\phi_y^{\sigma' n} \end{aligned}$$

Tree level evolution

1. we shift the IR field, $\hat{\phi}_x \to \hat{\Phi} + \hat{\phi}_x$, where the homogeneous component $\hat{\Phi} = (\Phi, \Phi)$ is at the minimum of the real part of the potential
2. we introduce the shifted functions, $\tilde{U}(\phi) = U(\Phi + \phi) - \phi^2 U''(\Phi)/2$, $\tilde{V}_x(\hat{\phi}, \hat{\phi}') = V_x(\hat{\Phi} + \hat{\phi}, \hat{\Phi} + \hat{\phi}')$
3. we solve EOM $\hat{D}^{-1}\hat{\varphi} = \hat{L}$, where$L^\sigma_x = \sigma\tilde{U}'(\phi^\sigma_x) - 2\int dy \partial_{\phi^\sigma_x}\tilde{V}_{x-y}(\hat{\phi}_x, \hat{\phi}_y)$
4. we substitute it back into the blocked action, $S_{k-\Delta k} = S_k + \Delta S_k$, which yields

$$\Delta S_k = \frac{\Delta k}{2}\int dx dy \hat{L}_x \hat{D}^{(k)}_{x-y}\hat{L}_y,$$

   with the propagator $\hat{D}^{(k)}_{x-y} = \int \frac{d^4q}{(2\pi)^4}\delta_{|\mathbf{q}|-k}\hat{D}_q e^{-i(x-y)q}$
5. projecting ΔS_k onto the ansatz results in the evolution equation

$$\dot{S} = -\frac{1}{2}\sum_{\sigma\sigma'}\int dx dy \tilde{U}'(\phi^\sigma_x)\sigma D^{(k)\sigma\sigma'}_{x-y}\sigma'\tilde{U}'(\phi^\sigma_y)$$

Renormalized trajectory

For each value of the gliding cutoff, k, the function h_q receives contributions from the region $k - \Delta k < |\mathbf{q}| < k$.
The tree-level evolution contributes to the bare action as $S = S_B + S_o$, where

$$S_o[\hat{\phi}] \quad = \quad \frac{1}{2} \sum_{\sigma,\sigma'} \sigma\sigma' \int dxdy \tilde{U}'(\phi^{\sigma}_x) D^{(k,\Lambda)\sigma\sigma'}_{x-y} \tilde{U}'(\phi^{\sigma'}_x),$$

where the propagator, covering the eliminated modes is given by

$$\hat{D}^{(k,\Lambda)}_{x-y} = \int \frac{d^4q}{(2\pi)^4} \Theta(|\mathbf{q}| - k)\Theta(\Lambda - |\mathbf{q}|)\hat{D}_q e^{-i(x-y)q}.$$

The kernel of the bilocal action turns out to be

$$h^{\sigma,\sigma'}_{m,n,x-y} = g_{m+1} g_{n+1} \sigma D^{(k,\Lambda)\sigma,\sigma'}_{x-y} \sigma'.$$

Its evolution is driven by the closed part of the action.

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