Solution of the $O(N)$ model in the $\mathcal{O}(\lambda^2)$ truncation of 2PI: an IR problem

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- Motivation
- Introduction
- The IR problem
- Conclusions

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Motivation

- Stability of the order of the phase transition wrt approximations
  Hartree-Fock: $1^{\rm st}$ ✘ 2-Loop: $2^{\rm nd}$ ✔ $\mathcal{O}(\lambda^2)$ **?**
- 2-Loop exponents are mean field $\to \mathcal{O}(\lambda^2)$ truncation has Z, might lead to non-mean field exponents
- Transverse gap mass in the 2-Loop strongly breaks the Goldstone theorem. Does it get better?
- IR problems in the 2-Loop $O(2)_\mu$: similar mechanics to what we will see here.

Introduction to 2PI

2PI is an exact, functional method which gives self-consistent equations for the 1- and 2-point function.

A bilocal source is introduced in the generating functional

$$Z[J,K] = e^{W[J,K]} = \int \mathcal{D}\varphi \, \exp\left[-S_0 - S_{\rm int} + \varphi \cdot J + \varphi \cdot K \cdot \varphi \right]$$

The 2PI effective action defined through a double Legendre transform

$$\gamma[\phi, G] = W[J,K] - \int d^4x \underbrace{\frac{\delta W[J,K]}{\delta J(x)}}_{\phi(x)} J(x) - \int d^4x \int d^4y \underbrace{\frac{\delta W[J,K]}{\delta K(x,y)}}_{[\phi(x)\phi(y)+G(x,y)]/2} K(x,y)$$

The physical $\bar{\phi}(x)$ and $\bar{G}(x,y)$ are determined from stationarity conditions at vanishing sources ($J, K \to 0$)

$$\left.\frac{\delta\gamma[\phi,G]}{\delta\phi(x)}\right|_{\bar{\phi}(x)} = 0, \qquad \left.\frac{\delta\gamma[\phi,G]}{\delta G(x,y)}\right|_{\bar{G}(x,y)} = 0$$

The 2PI effective action has a diagrammatic expansion, which needs to be truncated to be solved.

$\gamma[\phi, G]$ can be written as shown in Cornwall et al., PRD **10**, 2428 (1974)

$$\gamma[\phi, G] = S_0(\phi) + \frac{1}{2}\mathrm{Tr}\log G^{-1} + \frac{1}{2}\mathrm{Tr}\left[G_0^{-1}G - 1\right] + \gamma_{\mathrm{int}}[\phi, G]$$

S_0 is the free action,

G_0 is the free propagator,

$\gamma_{\mathrm{int}}[\phi, G]$ contains all the 2PI graphs constructed with vertices from $S_{\mathrm{int}}(\phi + \varphi)$.

The Tr is to be understood in all indices and as integration over coordinates.

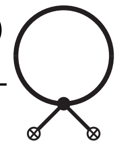
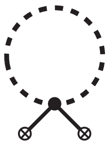
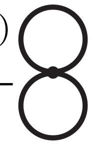
The 1PI effective action is recovered: $\Gamma_{\mathrm{1PI}}[\phi] = \gamma[\phi, \bar{G}]$.

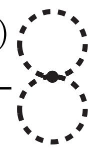
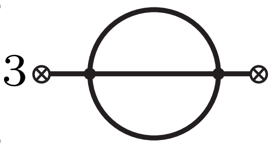
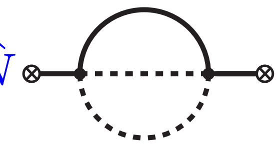
$\mathbf{O(N)}$ **model:** choosing the basis $\vec{\phi} = (\phi, 0, \ldots, 0)$ the propagator has the representation $G = \mathrm{diag}\,(G_L, G_T, \ldots, G_T)$.

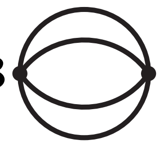
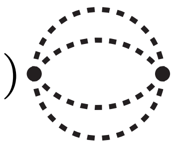
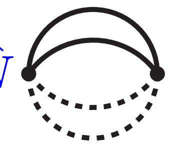
Equations

The 2PI effective potential, with $\hat{N} \equiv N - 1$ and $\lambda_{0,2}^{(\alpha A + \beta B)} \equiv \alpha\lambda_{0,2}^{(A)} + \beta\lambda_{0,2}^{(B)}$

$$\gamma[\phi, G_L, G_T] = \frac{1}{2}\mathrm{Tr}\int_Q^T \left[\log(G^{-1}(Q)) + G_0^{-1}(Q)\cdot G(Q)\right] + \frac{1}{2}m_2^2\phi^2 + \frac{\lambda_4\phi^4}{24N}$$

$+\dfrac{\lambda_2^{(A+2B)}}{12N}$  $+\dfrac{\lambda_2^{(\hat{N}A)}}{12N}$  $+\dfrac{\lambda_0^{(A+2B)}}{24N}$  $+\dfrac{\lambda_0^{(\hat{N}A)}}{12N}$ 

$+\dfrac{\lambda_0^{(\hat{N}^2A+2\hat{N}B)}}{24N}$  $-\dfrac{\lambda_\star^2}{36N^2}\Big[3$  $+\hat{N}$  $\Big]$

$-\dfrac{\lambda_\star^2}{144N^2}\Big[3$  $+(N^2-1)$  $+2\hat{N}$  $\Big].$

The field and gap equations are derived then as

$$0 = \left.\frac{\delta\gamma[\phi, G_L, G_T]}{\delta\phi}\right|_{\bar{\phi},\bar{G}_L,\bar{G}_T} = \left.\frac{\delta\gamma[\phi, G_L, G_T]}{\delta G_L}\right|_{\phi,\bar{G}_L,\bar{G}_T} = \left.\frac{\delta\gamma[\phi, G_L, G_T]}{\delta G_T}\right|_{\phi,\bar{G}_L,\bar{G}_T}.$$

And the curvature masses are defined as

$$\hat{M}_L^2 = 4\bar{\phi}^2\gamma''(\bar{\phi}^2) + 2\gamma'(\bar{\phi}^2) = 4\bar{\phi}^2\left.\frac{df(\phi)}{d\phi}\right|_{\bar{\phi}} + 2f(\bar{\phi}), \qquad \hat{M}_T^2 = 2\gamma'(\bar{\phi}^2) = 2f(\bar{\phi}),$$

with $\gamma(\phi^2) := \gamma[\phi, \bar{G}_L, \bar{G}_T]$ and $f(\phi) := \left.\dfrac{1}{\phi}\dfrac{\delta\gamma[\phi, G_L, G_T]}{\delta\phi}\right|_{\bar{\phi},\bar{G}_L,\bar{G}_T}$

Renormalization

Renormalization is similar to that of Markó et al., PRD **87** 105001 (2013). See also Berges et al., Annals Phys. **320** 344 (2005).

- **Prescriptions on 2- and 4-point functions,** at $T = T_\star$ and $\bar{\phi} = 0$.
- Truncation artefact: ambigous n-point functions require more counterterms.
- 3 renormalization + 6 consistency conditions (few of them are trivial) **fix 9 counterterms**.
- Only **2 renormalized parameters:** $\mathrm{m}_\star^2$, $\boldsymbol{\lambda}_\star$ and a renormalization scale $T_\star$.
- Counterterms are **temperature independent**, that is they are the same at any T.
- Compared to the 2-Loop case, there is a need for wave-function renormalization.
- Triviality of the theory is seen through the appearance of the Landau pole, Λ_p. For $\Lambda > \Lambda_\mathrm{p}$ the theory becomes unstable.

Numerics

We solve the coupled field and gap equations iteratively in Euclidean space.

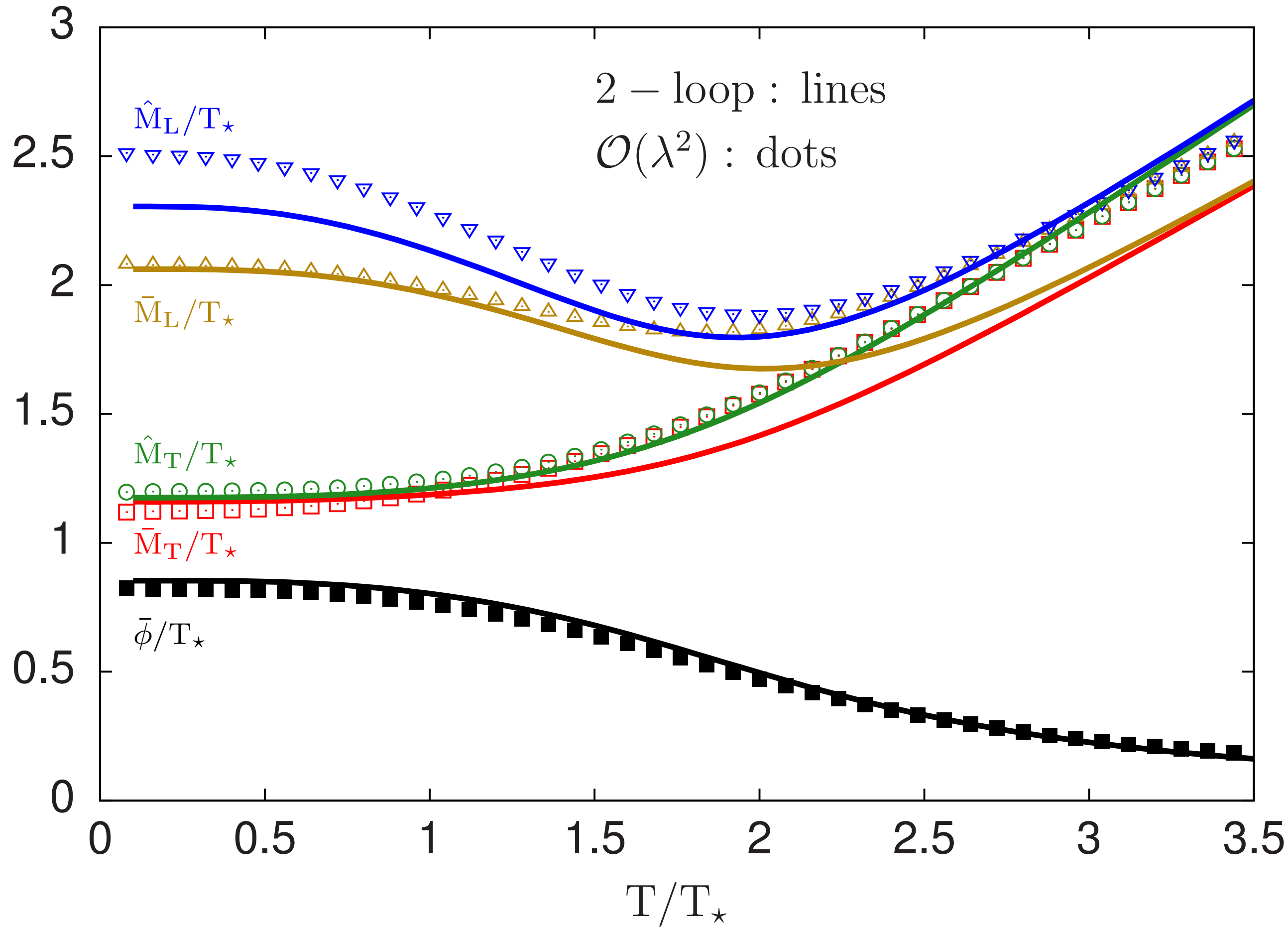
We discretize the propagators on a $N_\tau \times N_s$ grid:

$$\omega_n = 2\pi n T,\; n \in [0..N_\tau - 1], \text{ and } k = (s+1)\frac{\Lambda}{N_s},\; s \in [0..N_s - 1].$$

- Rotation invariance $\Rightarrow$ only 1D in momentum space.
- Convolutions are done using FFT routines.
- Moderate cutoff values are used as both Λ/N_s and Λ^3/N_τ has to be small.
- Numerical method was developed in Markó et al., PRD **86** 085031 (2012).

Light mesons in the $\mathcal{O}(\lambda^2)$ truncation $(\mathrm{N}=4)$

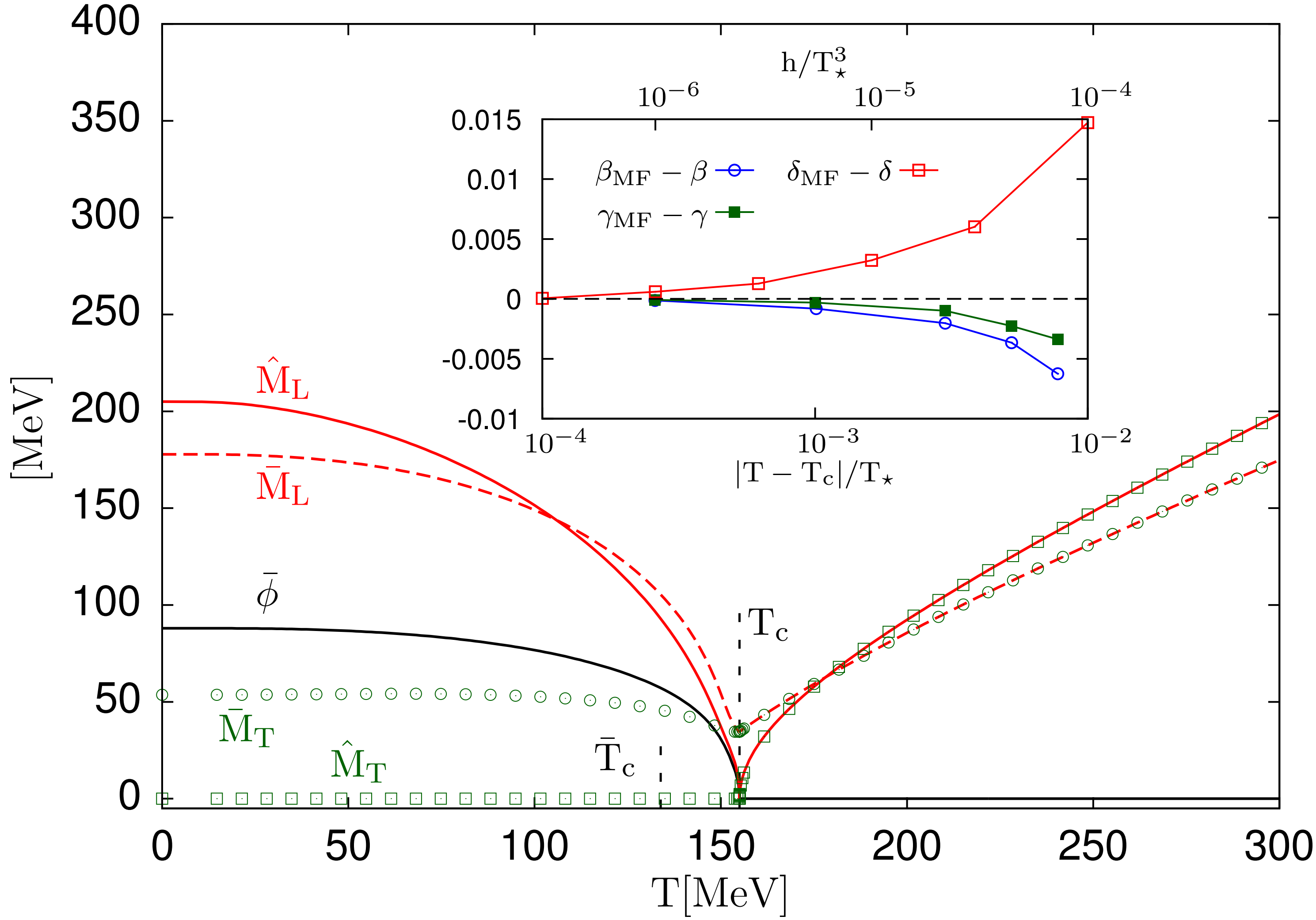
Physical parametrization requires relatively large external source (h) values, to accomodate for $\hat{M}_T \approx m_\pi$.



- High temperature: $\hat{M} \approx \bar{M}$ only in the $\mathcal{O}(\lambda^2)$ truncation.
- Low temperature: Only $\hat{M}_L$ differs strongly and $\bar{M}_T/\hat{M}_T \lesssim 1$.

The IR problem

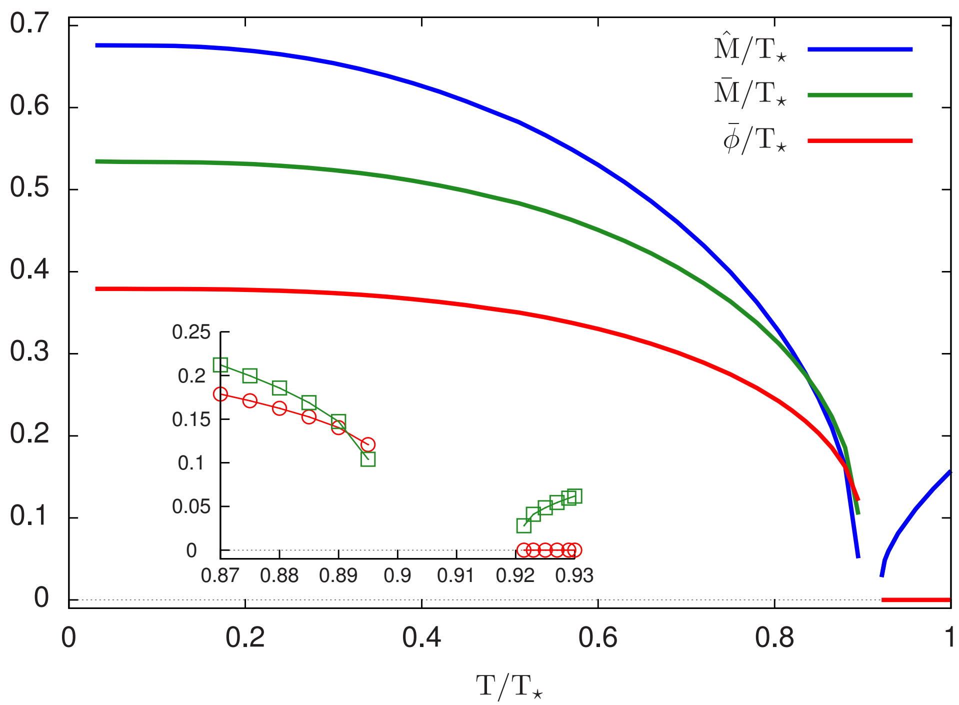
Chiral limit ($h \to 0$)? Expectations set by looking back at the 2-Loop results.



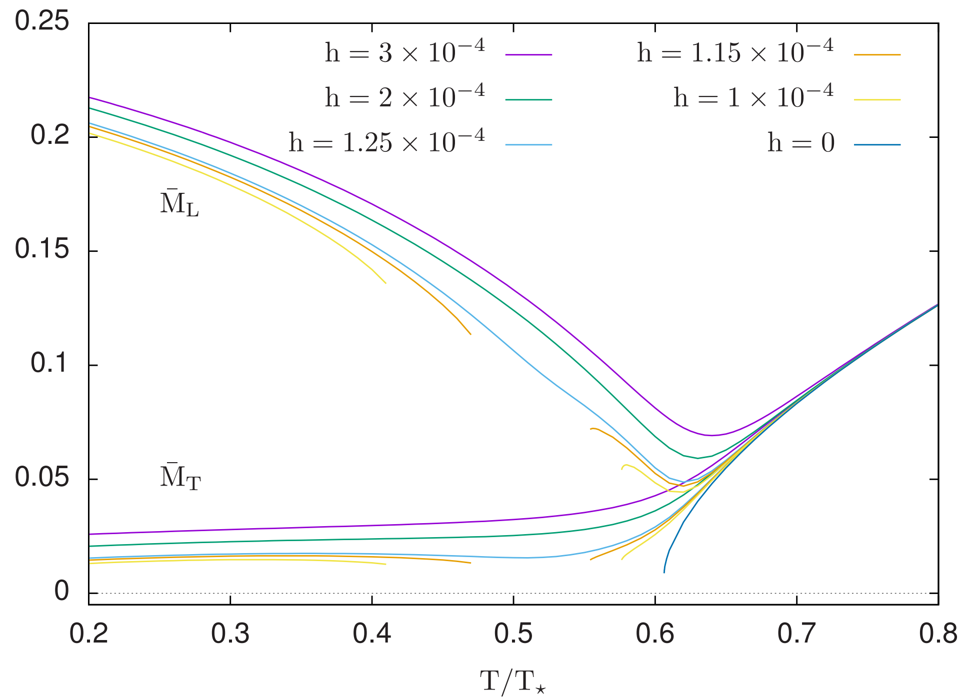
- 2^{nd} order PT.
- Mean field exponents.
- Goldstone theorem is only fulfilled by $\hat{M}_T$.

The IR problem

Chiral limit ($h = 0$) in the $\mathcal{O}(\lambda^2)$ truncation:



$N = 1$



$N = 4$

- Low h: temperature range, with **NO** solution.
- Chiral limit: T_c is missing, the gap engulfs it.

Flashback: 2-Loop $\mathrm{O}(2)$ at finite μ

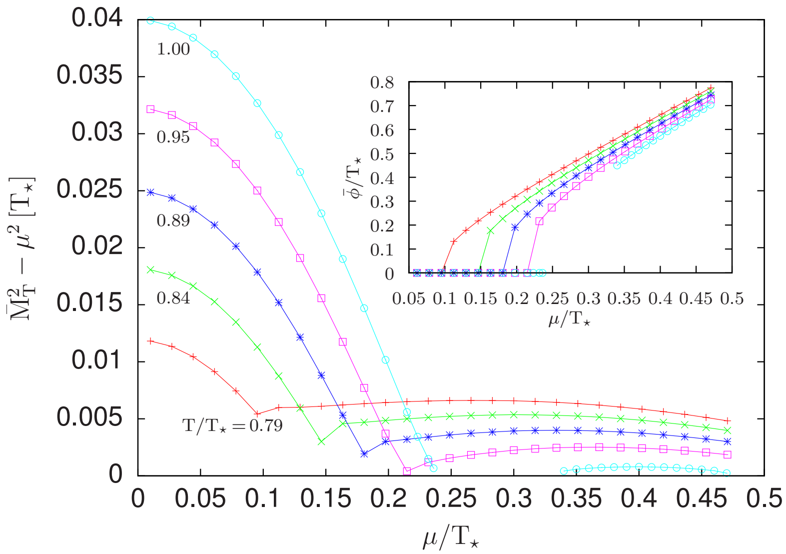
Loss of solution

We define $\bar{\mu}_c(T)$ as

$$\bar{M}^2_{\phi=0,T,\mu=\bar{\mu}_c(T)} = \bar{\mu}_c^2\,,$$

which is the inverse of $\bar{T}_c(\mu)$.

- $\mu > \bar{\mu}_c(T) \to$ no solution for gap eq at $\phi = 0$.
- $\phi_c(\mu, T)$: the smallest ϕ for which a solution of the gap equations exists.
- Solution of the coupled gap and field equations is lost when: $\bar{\phi}(\mu, T) < \phi_c(\mu, T)$.



Markó et al., PRD **90** 125021 (2014)

Localized 2PI equations: a useful tool

- Idea previously used in e.g. M. Bordag and V. Skalozub, J. Phys. A 34, 461 (2001) and U. Reinosa and Zs. Szép, Phys. Rev. D 85, 045034 (2012).

- For light modes (small masses) diagrams are dominated by the $Q = 0$ part of the propagators.

- Approximate the non-local self-energy with its $Q = 0$ part, using the gap equations at $Q = 0$.

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- Approximate the non-local self-energy with its $Q = 0$ part, using the gap equations at $Q = 0$.

1. Take the coupled set of the (finite) field and gap equations.

2. Compute the diagrams with the ansatze $\bar{G}^{-1}_{L,T}(Q) = Q^2 + \bar{M}^2_{L,T}$, that is tree-level type propagators.

3. Leads to more analytical control (e.g. through HTE) and/or faster numerics.

How do we define the finite localized equations? The original counterterms do not renormalize the local equations.

Localized 2PI

- $N = 1$ gap equation needs more counterterms, but **can be renormalized to all orders.**

- Results in using **the rule:** replace bare parameters with renormalized ones + replace integrals with their finite versions.

- $N = 1$ field equation OR $N = 4$ coupled gap equations **lead to contradictions.** No constructive way to renormalize.

- However the $N = 1$ gap equation rule is the natural way to define the finite equations.

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- $N = 1$ field equation OR $N = 4$ coupled gap equations **lead to contradictions.** No constructive way to renormalize.
- However the $N = 1$ gap equation rule is the natural way to define the finite equations.

The resulting localized equations:

$$\bar{M}_L^2 = m_\star^2 + \frac{\lambda_\star}{2N}\Big(\phi^2 + \mathcal{T}_{\rm F}[\bar{G}_L]\Big) + \hat{N}\frac{\lambda_\star}{6N}\mathcal{T}_{\rm F}[\bar{G}_T] - \frac{\lambda_\star^2\phi^2}{18N^2}\Big(9\mathcal{B}_{\rm F}[\bar{G}_L] + \hat{N}\mathcal{B}_{\rm F}[\bar{G}_T]\Big)$$

$$-\frac{\lambda_\star^2}{18N^2}\Big(3\mathcal{S}_{\rm F}[\bar{G}_L] + \hat{N}\mathcal{S}_{\rm F}[\bar{G}_L;\bar{G}_T;\bar{G}_T]\Big)\,,$$

$$\bar{M}_T^2 = m_\star^2 + \frac{\lambda_\star}{6N}\Big(\phi^2 + \mathcal{T}_{\rm F}[\bar{G}_L]\Big) + (N+1)\frac{\lambda_\star}{6N}\mathcal{T}_{\rm F}[\bar{G}_T] - \frac{\lambda_\star^2\phi^2}{9N^2}\mathcal{B}_{\rm F}[\bar{G}_L;\bar{G}_T]$$

$$-\frac{\lambda_\star^2}{18N^2}\left(\mathcal{S}_{\rm F}[\bar{G}_T;\bar{G}_L;\bar{G}_L] + (N+1)\mathcal{S}_{\rm F}[\bar{G}_T]\right)\,,$$

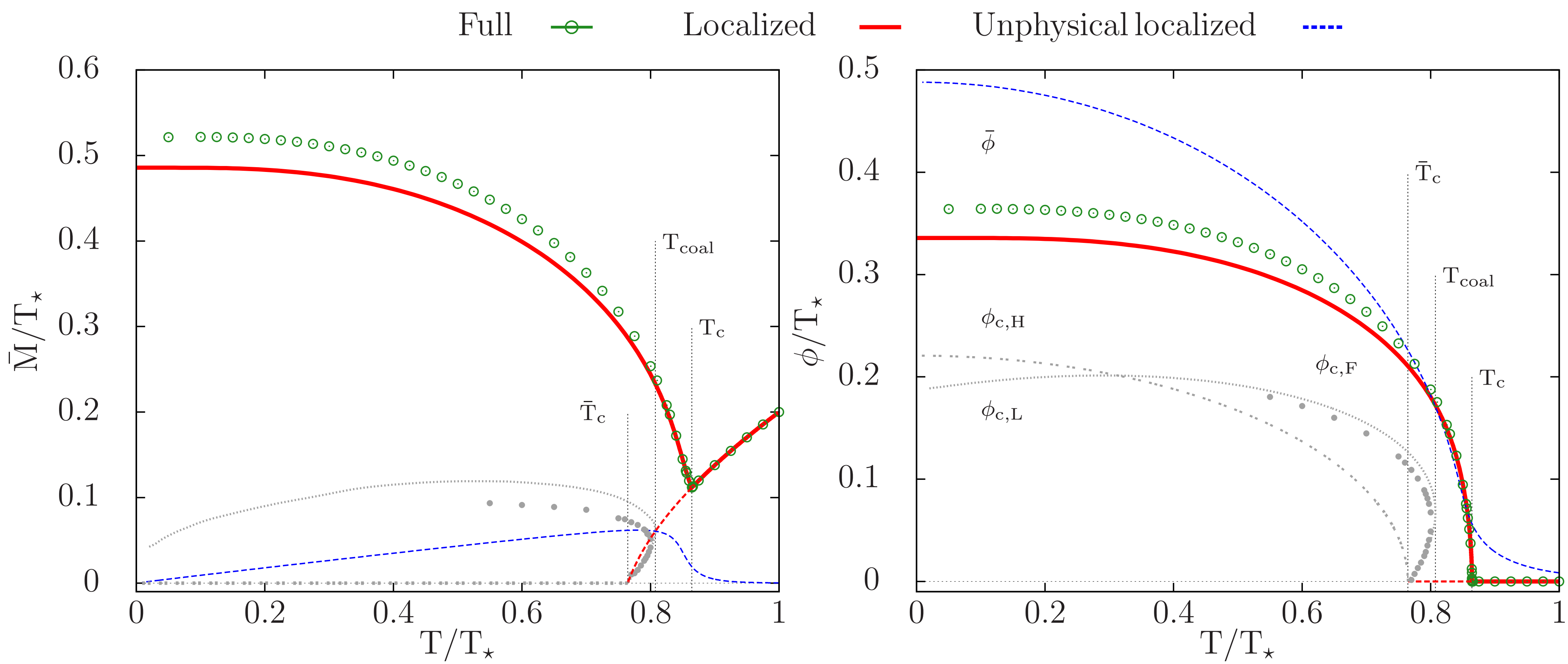
$$\frac{h}{\bar{\phi}} = m_\star^2 + \frac{\lambda_\star}{6N}\bar{\phi}^2 + \frac{\lambda_\star}{2N}\mathcal{T}_{\rm F}[\bar{G}_L] + \hat{N}\frac{\lambda_\star}{6N}\mathcal{T}_{\rm F}[\bar{G}_T] - \frac{\lambda_\star^2}{18N^2}\Big(3\mathcal{S}_{\rm F}[\bar{G}_L] + \hat{N}\mathcal{S}_{\rm F}[\bar{G}_L;\bar{G}_T;\bar{G}_T]\Big)\,.$$

Localized 2PI

To remain close to our original renormalization prescription, we do subtractions at $T_\star$:

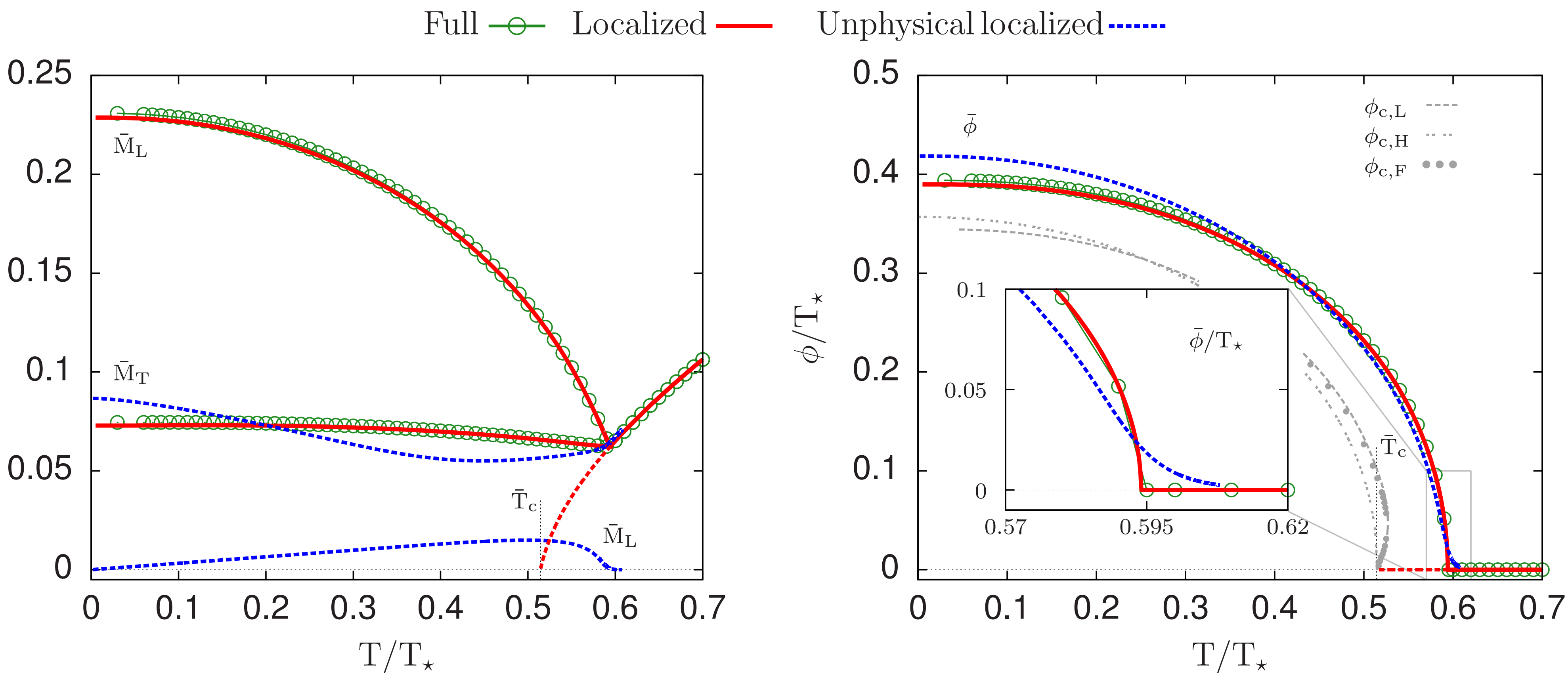
$$
\begin{aligned}
\mathcal{T}_{\rm F}[\bar{G}] &\equiv \mathcal{T}[\bar{G}] - \mathcal{T}_\star[G_\star] - (\bar{M}^2 - m_\star^2)\frac{d\mathcal{T}_\star[G_\star]}{dm_\star^2}\,, \\
\mathcal{B}_{\rm F}[\bar{G}] &\equiv \mathcal{B}[\bar{G}] - \mathcal{B}_\star[G_\star]\,, \\
\mathcal{B}_{\rm F}[\bar{G}_L;\bar{G}_T] &\equiv \mathcal{B}[\bar{G}_L;\bar{G}_T] - \mathcal{B}_\star[G_\star]\,, \\
\mathcal{S}_{\rm F}[\bar{G}] &\equiv \mathcal{S}[\bar{G}] - \mathcal{S}_\star[G_\star] - (\bar{M}^2 - m_\star^2)\frac{d\mathcal{S}_\star[G_\star]}{dm_\star^2} - 3\mathcal{T}_{\rm F}[\bar{G}]\mathcal{B}_\star[G_\star]\,, \\
\mathcal{S}_{\rm F}[\bar{G}_L;\bar{G}_T;\bar{G}_T] &\equiv \mathcal{S}[\bar{G}_L;\bar{G}_T;\bar{G}_T] - \mathcal{S}_\star[G_\star] - (2\mathcal{T}[\bar{G}_T] + \mathcal{T}[\bar{G}_L])\mathcal{B}_\star[G_\star] \\
&\quad -\frac{1}{3}\left[2(\bar{M}_{\rm T}^2 - m_\star^2) + \bar{M}_{\rm L}^2 - m_\star^2\right]\frac{d\mathcal{S}_\star[G_\star]}{dm_\star^2}\,.
\end{aligned}
$$

Check, using the 2-Loop results, $\mathrm{N} = 1$:

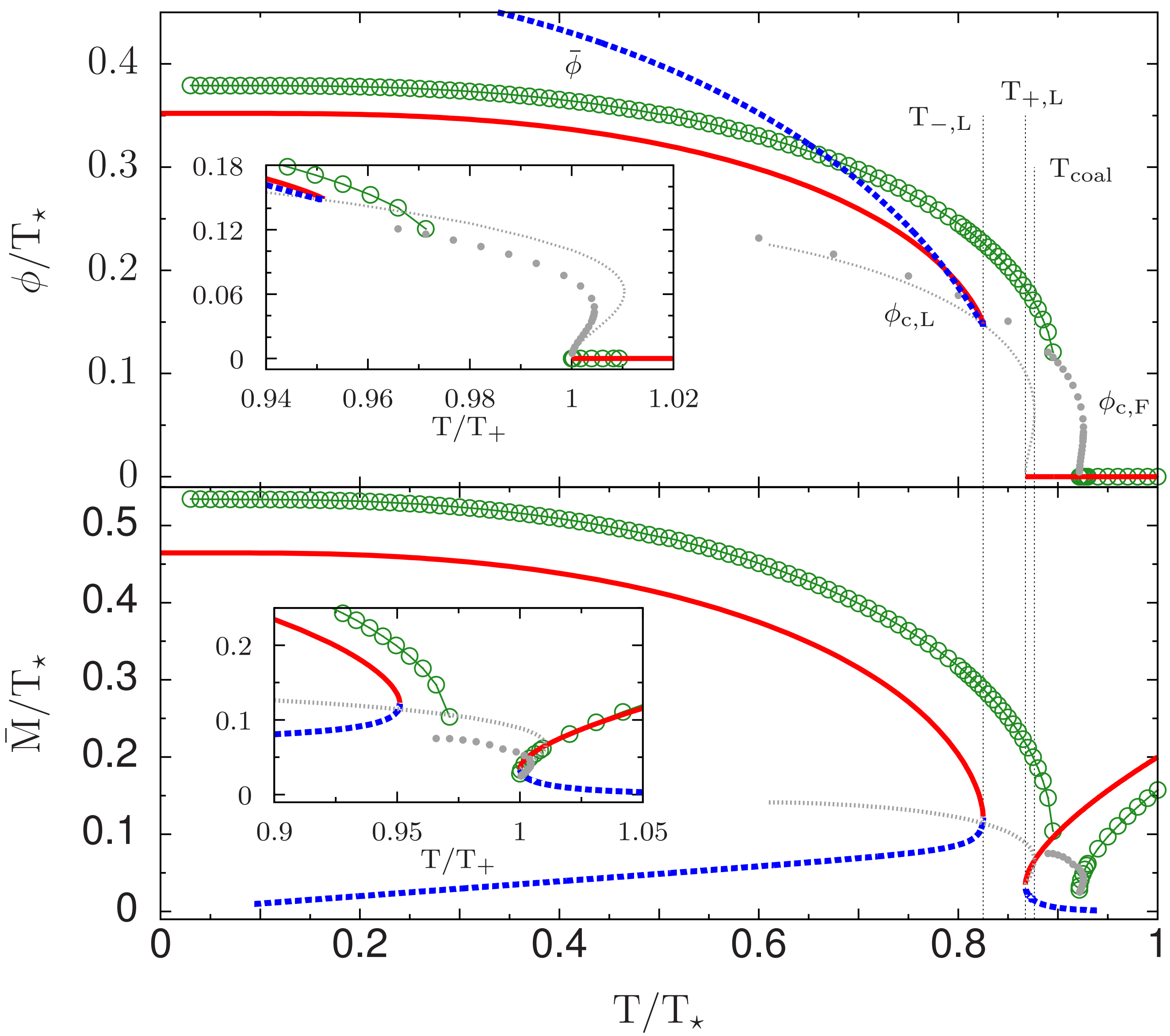


- Localized solution agrees quite well with the full one.
- ϕ_c curves delimit regions where the gap equation has no solution.
- Localized equations have an unphysical solution $\rightarrow$ we cannot rule it out in the full, iterative method is not decisive.

Check, using the 2-Loop results, $\mathrm{N} = 4$:

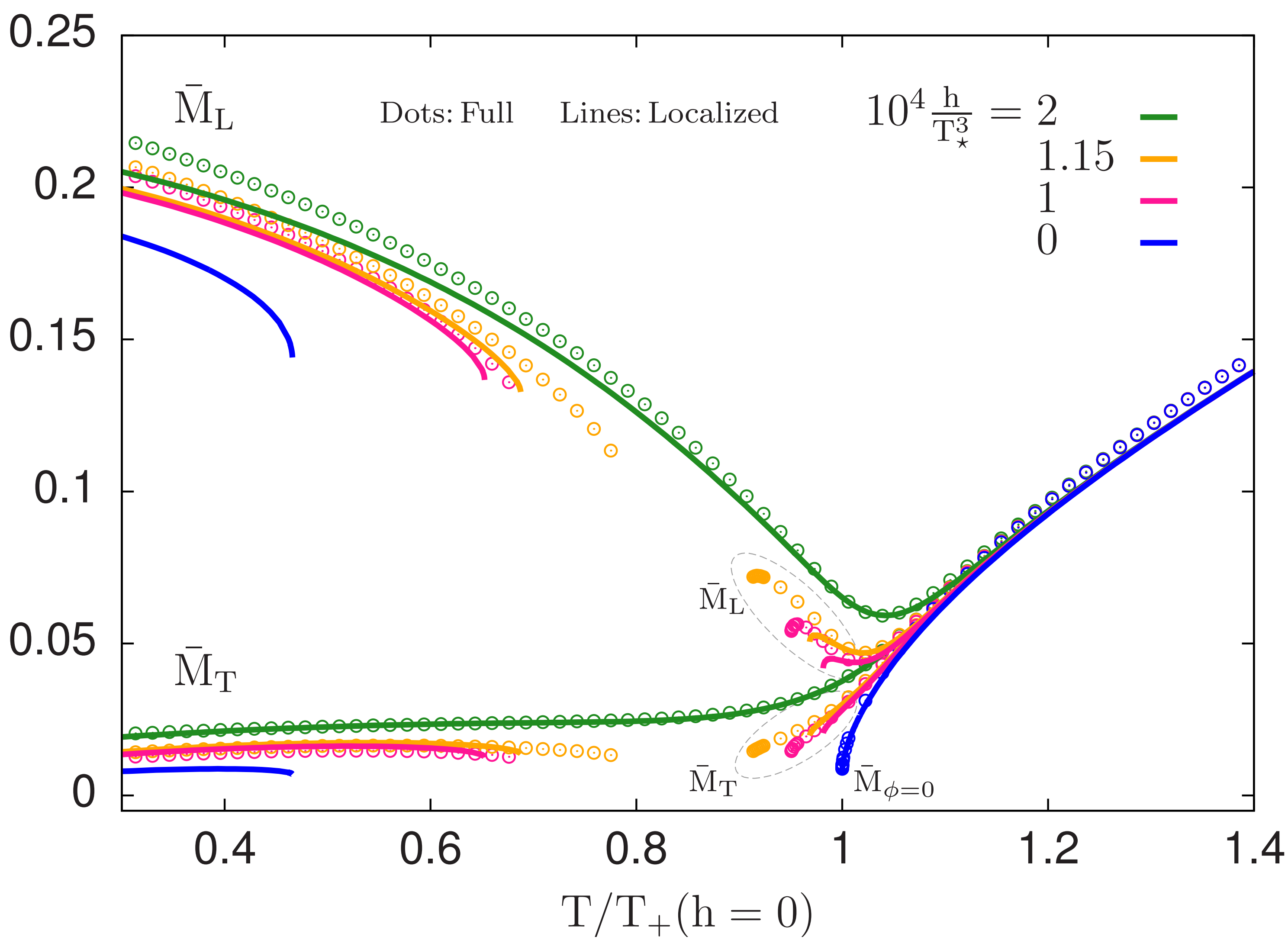


Comparison in $\mathcal{O}(\lambda^2)$, $\mathbf{N = 1}$



- ϕ_c curve meets corresponding $\bar{\phi}$ curve.
- Unphysical and physical solutions merge.
- Would-be T_c is in the temperature gap.
- $T_{-/+}$ are defined as the lower/higher end-points of the gap.

Comparison in $\mathcal{O}(\lambda^2)$, $\mathbf{N} = 4$



- As h is lowered the temperature gap appears at the smallest $\bar{M}_T$ values.
- Localized unphysical solutions are found, but not plotted here.

What can we say analytically?

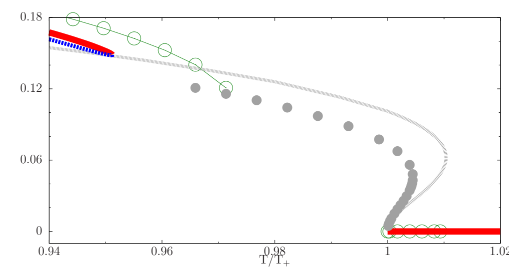
Using HTE sheds some light on what is happening ($N = 1$ case, to keep things simple):

- Assuming there is a $T_c: \ \bar{M}(T_c) = \bar{\phi}(T_c) = 0$, and the following equation is satisfied

$$0 = m_\star^2 + \frac{\lambda_\star}{2}\mathcal{T}_{\rm F}^{T_{\rm c}}[\bar{G}_{\rm c}] - \frac{\lambda_\star}{6}\mathcal{S}_{\rm F}^{T_{\rm c}}[\bar{G}_{\rm c}]\,, \qquad \mathcal{S}[\bar{G}] \sim -T^2 \log\frac{\bar{M}^2}{T^2}$$

  However $\mathcal{S}_{\rm F}^{T_{\rm c}}[\bar{G}_{\rm c}]$ is IR divergent $\Rightarrow$ the equation is meaningless.

- The whole equation decreases as $M \to 0 \Rightarrow$ at some temperature the $\phi = 0$ solution will be lost: T_+.

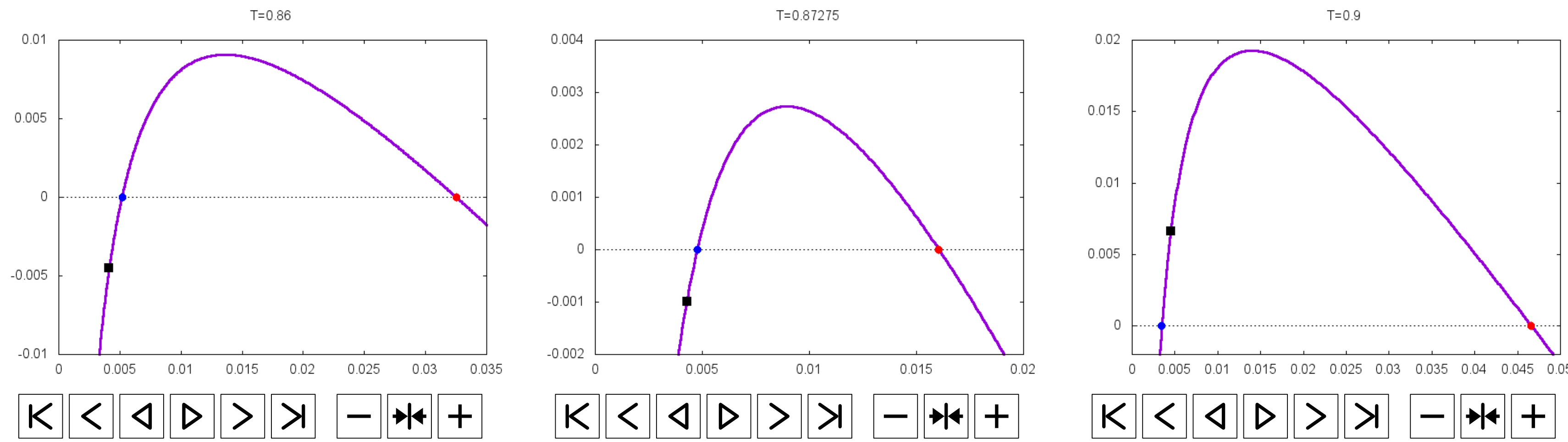


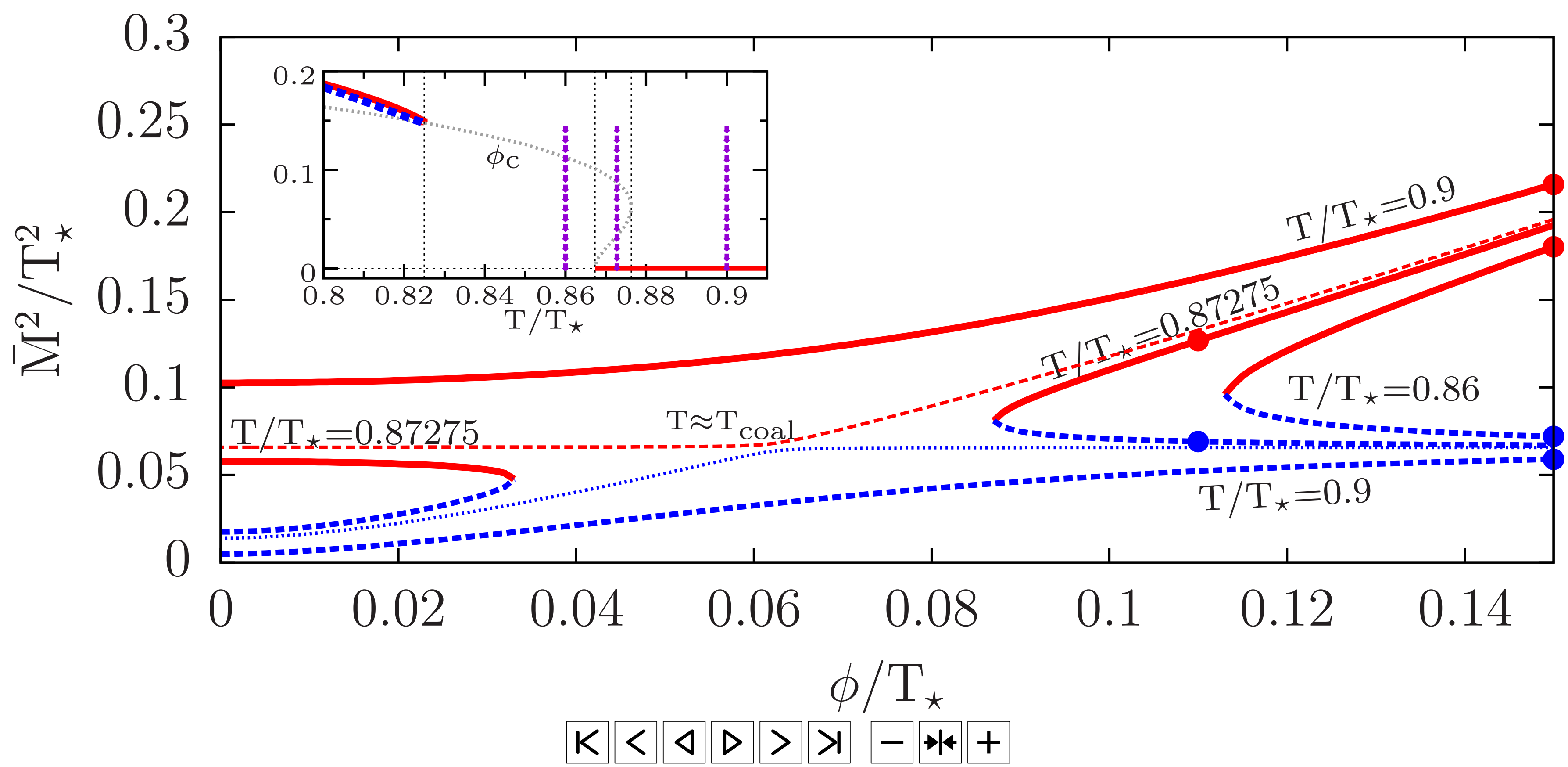
- Approaching from the broken phase one has (combining the gap and field equations)

$$\bar{\phi}^2 = -\frac{6\bar{M}^2}{3\lambda_\star^2\mathcal{B}_{\rm F}[\bar{G}] - 2\lambda_\star}\,, \qquad B[\bar{G}] \sim \frac{T}{\bar{M}}$$

  which turns negative at some point signaling, that the broken phase solution must cease to exist at some temperature: T_-.

What more can we say numerically?





Conclusions

From full 2PI

- The gap equation(s) at fixed $T < T_{\rm coal}$ has no solution for a range of ϕ.
- $T_{-/+}$ are limiting temperature values above/below which $\bar{\phi}$ enters the restricted ϕ-region.
- The 2-Loop also had the restricted ϕ-region, $\bar{\phi}$ never entered it though.
- Whether $\bar{\phi}$ is engulfed can be controlled by many parameters: $T, h, \mu, \ldots$

From localization

- The shape of the curves suggest similar behaviour.
- We could not find unphysical solutions in the full 2PI.
- But we could not find them iteratively in the localized approx. either.

What we learned

× Both approximations miss an anomalous dimension.

× Therefore IR divergences are not tamed.

× Could be corrected by higher orders (similarly as in the 2-Loop).

× Vertex resummation needed, e.g. NLO $1/N$.