Fixed Points Structure & Effective Fractional Dimension for O(N) Models with Long-Range Interactions

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Overview

1. Known Results
   - 1d Long Range Ising model
   - Long Range Ising model
2. Functional Renormalization group approach
3. Effective Dimension
4. In Conclusion
   - short range results

Ising system with long range interactions

Why?

- The Ising model is a paradigmatic model in statistical theory, however in case of long range interactions there are still many details about the critical properties which remains unknown.
- New Monte Carlo (MC) results led to renewed interest on this problem.
- Long range interactions are widely present in neural system modeling and in spin glass physics.

Long Range Hamiltonian

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Effective field theory

$$\int d^d x \left\{ -\frac{Z_\sigma}{2}\phi(x)\Delta^{\sigma/2}\phi(x) - \frac{Z_2}{2}\phi(x)\Delta\phi(x) + \ldots + U(\phi(x)) \right\}$$

Long Range Interactions in 1d system

Four regimes

- $\sigma < \mathbf{0}$ $T_c = \infty$ **Ill defined model**.
- $\mathbf{0} < \sigma < \mathbf{1/2}$ **Mean field exponents** ($\eta = 2 - \sigma$ and $\nu = \frac{1}{\sigma}$).
- $\mathbf{1/2} < \sigma < \mathbf{1}$ **Long range behavior** ($\eta = f(\sigma)$ and $\nu =?$).
- $\sigma > \mathbf{1}$ **No phase transition** ($T_c = 0$).[a]

[a] S.A. Cannas PRB 52,5

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Special Case

When $\sigma = 1$ it has a Kosterlitz Thouless transition with discontinuity in the magnetization value at T_c

Long range interactions in d dimensions

Traditional Results

Three regimes:

- **0** $<\sigma<$ **d/2 Mean field exponents** ($\eta = 2-\sigma$ and $\nu = \sigma^{-1}$).
- **d/2** $<\sigma<$ **2 Long range exponents** ($\eta = f(\sigma)$ and $\nu = ?$).
- $\sigma >$ **2 Short range exponents** ($\eta = \eta_{SR}$ and $\nu = \nu_{SR}$).[a]

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Peculiar Long Range Behavior

Using ϵ-expansion technique with $\epsilon = 2\sigma - d$ and $1/n$ expansion is possible to calculate the critical exponents

$$\eta = 2 - \sigma + O(\epsilon^3)$$

This result was believed to be exact at any order in ϵ leading to η to stick at the classical result with a **discontinuity** in $\sigma = 2$.

Removal of the discontinuity

Sak's Results

The anomalous dimension cannot be less than η_{SR},

$$\eta = 2 - \sigma \quad \sigma < \sigma^*$$
$$\eta = \eta_{SR} \quad \sigma > \sigma^*$$

where $\sigma* = 2 - \eta_{SR}$ and **no discontinuity** is present in η values.

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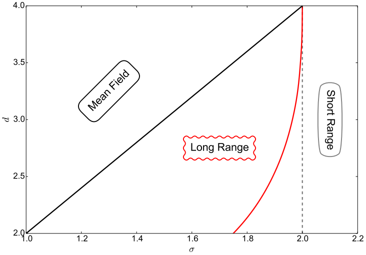
System regimes



Monte Carlo Results: Controversy

Luijte and Blote[a] results (2002) seemed to confirm Sak results, but new, more complete, results (2013)[b] question on Sak validity

[a] E. Luijte & H.W. Blote PRL 89, 025703
[b] M. Picco, arXiv:1207.1018

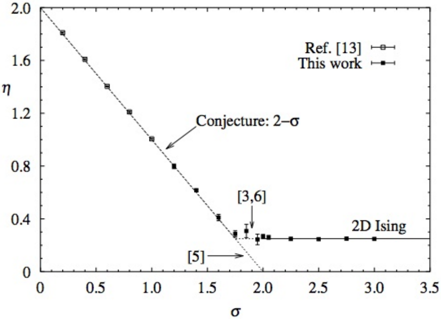


Figure: MC 2002

VS

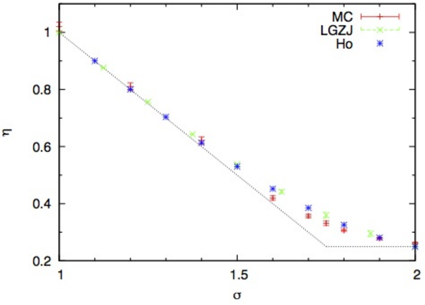


Figure: MC 2013

Effective average action flow

Taking the derivative of the Gibbs free energy definition with respect to the renormalization time $t = -\log(\frac{k}{\Lambda})$, We get:

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Effective Average Action Flow Equation

$$\partial_t \Gamma_t[\phi] = \int \frac{d^d q}{(2\pi)^d} \partial_t R_t(q) \left(\Gamma_t^{(2)}(q, -q) + R_t(q)\right)^{-1} \quad (1)$$

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1. $\partial_{\mathbf{t}} \mathbf{R_t(q)}$ is the scale derivative of the cutoff.
2. $\boldsymbol{\Gamma}_{\mathbf{t}}^{(2)}(\mathbf{q}, -\mathbf{q})$ is the Fourier transform of the second functional derivative of the Gibbs free energy.
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The functional form of the flow equation evidences its one loop form

$$\partial_{\mathbf{t}}\mathbf{\Gamma_k}[\phi] = \bar{\partial}_{\mathbf{t}}\mathbf{Tr}\log\left(\mathbf{\Gamma_t^{(2)}} + \mathbf{R_t}\right)$$

Propagator Flow

Effective action ansatz

$$\int d^d x \left\{ -\frac{Z_\sigma}{2}\phi(x)\Delta^{\sigma/2}\phi(x) - \frac{Z_2}{2}\phi(x)\Delta\phi(x) + U(\phi(x)) \right\}$$

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Anomalous dimension correction?

$$\frac{\partial_{\mathbf{t}} \mathbf{Z}_\sigma}{\mathbf{Z}_\sigma} = \delta\eta$$

From the vertex flow equations we get:

$$\partial_{\mathbf{t}} \mathbf{Z}_\sigma = \mathbf{0} \rightarrow \delta\eta = \mathbf{0}$$

we retrieve LPA result which confirms Sak calculations

Effective Dimension

Definition of effective dimension

From the simplest approximations level we get two relations for the effective dimension of a Long Range model:

$$(Z_\sigma = 1, \quad Z_2 = 0) \rightarrow D_{eff} = \frac{2d}{\sigma}$$
$$(Z_\sigma = Z_t, \quad Z_2 = 0) \rightarrow D_{eff} = \frac{(2-\eta_{SR})d}{\sigma}.$$

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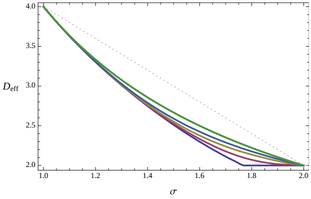
The two guess for the effective dimension



Coupling expansion

We know that in a short range model only the dimension plays a role in the critical properties of a model, but what about a long range model?

$$\int d^d x \left\{ \frac{Z_t}{2} |\nabla^{\sigma/2} \phi(x)|^2 + \frac{g_2}{2} \phi(x)^2 + \frac{g_4}{24} \phi(x)^4 + \cdots \right\}$$

Dimensional Argument

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Coupling dimension

$$[g_{2i}] = (i-1)D - 2i \; SR \quad \rightarrow \frac{D}{2} < \frac{i}{i-1}$$
$$[g_{2i}] = (i-1)d - \sigma i \; LR \quad \rightarrow \frac{d}{\sigma} < \frac{i}{i-1}$$

where the last is the condition for a coupling to be relevant.

Qualitative Description

I Approximation Level: No anomalous dimension

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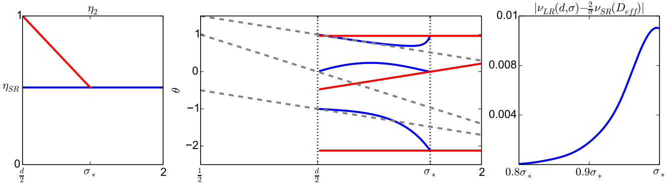
III Approximation Level: Mixed theory space

Competition between Short and Long range fixed points: ~~$\boldsymbol{D_{eff}}$~~

Fixed Points Solutions and Stability



Short Range potential flow

The fixed point potential

$$d\bar{U}(\bar{\rho}) - (d-2+\eta)\bar{\rho}\bar{U}^{(1)}(\bar{\rho}) - \frac{s_d}{d}\frac{1}{1+\bar{U}^{(1)}(\bar{\rho})+2\bar{\rho}\bar{U}^{(2)}(\bar{\rho})} = 0,$$

where $\bar{\rho} = \frac{\bar{\phi}^2}{2}$ and $\bar{\rho}_0$ is the potential minimum

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Critical exponents

$$\eta=\frac{s_d}{d}\frac{4\rho_0\bar{U}^{(2)}(\bar{\rho}_0)^2}{(1+2\bar{\rho}_0U^{(2)}(\bar{\rho}_0))^2},\qquad \nu=y^{-1},$$

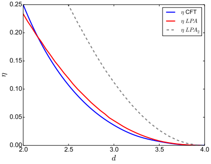
where $\bar{\rho}=\frac{\bar{\phi}^2}{2}$ and $\bar{\rho}_0$ is the potential minimum

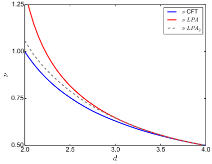
Universality Classes Long Range Ising

A. Codello J. Phys. A: Math. Theor. 45 (2012) 465006

Short Range Results

Ising Model





S. El-Showk, et al. Phys. Rev. Lett. 112, 141601 (2014),
A. Codello, N. Defenu, G. D'Odorico Phys. Rev. D 91, 105003 (2015)

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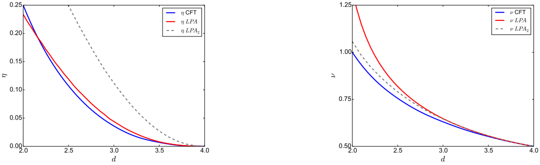
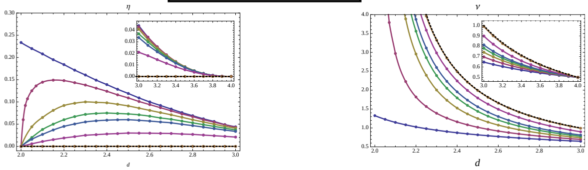


O(N) Models



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Quantitative Results

Short Range Corrections

Short Range corrections spoil dimensional equivalence. Small everywhere but at $\sigma \simeq \sigma_*$.

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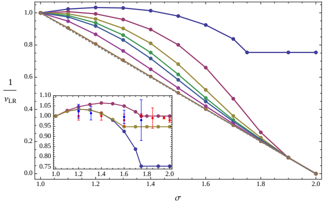
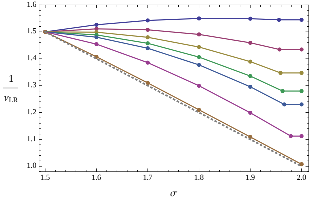
Correlation Length Exponent





Acknowledgements

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arXiv:1409.8322 [cond-mat.stat-mech]

Thank You

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How is this equation changed in the long range case?

$$\mathbf{D\bar{U}_t(\bar{m}) - \frac{D-2}{2}\bar{m}\bar{U}^1(\bar{m}) - \frac{s_D}{D}\frac{1}{1+\bar{U}^{(2)}(\bar{m})} = 0}$$

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Only the propagator dimension changes.

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As expected the presence of long range interactions modifies the coefficients of the flow equation in such a way to create an effective dimension greater than the real one. Blanchard et al suggest the relation $D = 4 + d - 2\sigma$