Negative Binomial Coherent States

Superstatistics of Quanta

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Our Goal Is

- to construct particular Coherent States
- designed to describe NB n distribution
- and non-extensive Tsallisean statistical weights

Outline

1. Statistical Weight from n-distribution
2. Nonlinear coherent states
3. Examples

Outline

Summary of ideal reservoir fluctuations 1

T.S.Biró et.al. Entropy 16 (2014) 6497, arxiv:1409.5975

Ideal gas formula and thermodynamical limit:

$$w_E^{\mathrm{LIM}}(\omega) = \lim_{\substack{n \to \infty \\ E \to \infty \\ E/n = T}} \left(1 - \frac{\omega}{E}\right)^n = \mathrm{e}^{-\omega/T}. \tag{1}$$

Poisson average on n at fix E:

$$w_E^{\mathrm{POI}}(\omega) = \mathrm{e}^{-\langle n \rangle \omega / E}. \tag{2}$$

Negative binomial (NB) average:

$$w_E^{\mathrm{NBD}}(\omega) = \left(1 + \frac{\langle n \rangle}{k+1} \frac{\omega}{E}\right)^{-(k+1)}. \tag{3}$$

Summary of ideal reservoir fluctuations 2

indico.kfki.hu/category/28

Expanding up to variance in n we obtain for general n fluctuations

Exact for Poisson, Bernoulli, Negative Binomial

$$T = \frac{E}{\langle n \rangle}, \quad \text{and} \quad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2} \tag{4}$$

$$w_E(\omega) \;=\; \left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} \tag{5}$$

In general T and q are related to expectation values of derivatives of the EoS $S(E)$ over reservoir fluctuations.

Summary of general reservoir fluctuations

Einstein: phase space $\Omega(E) = e^{S(E)}$

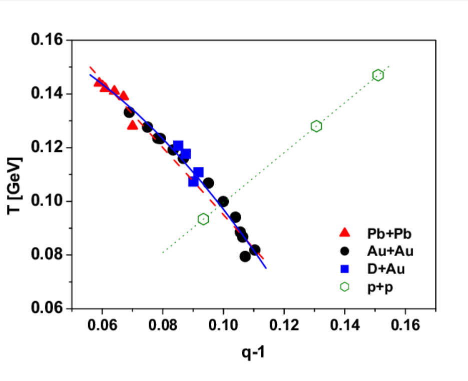
S(E) and its derivatives give

$$\frac{1}{T} = \langle\, S'(E)\, \rangle, \quad \text{and} \quad q = 1 + \frac{\Delta T^2}{T^2} - \frac{1}{C} \tag{6}$$

with $C = dE/dT$ total heat capacity and $S''(E) = -1/CT^2$.

Trends in Tsallis fits to data

G. Wilk's collection, Erice School on Complexity, 2015



Fits: $T_{AA} = 0.22 - 1.25(q-1)$ GeV; $T_{pp} = (q-1)$ GeV.

AA: assume constant variance, $\Delta T^2/T^2 = \sigma^2$ and ideal $C = E/T$. Then we obtain

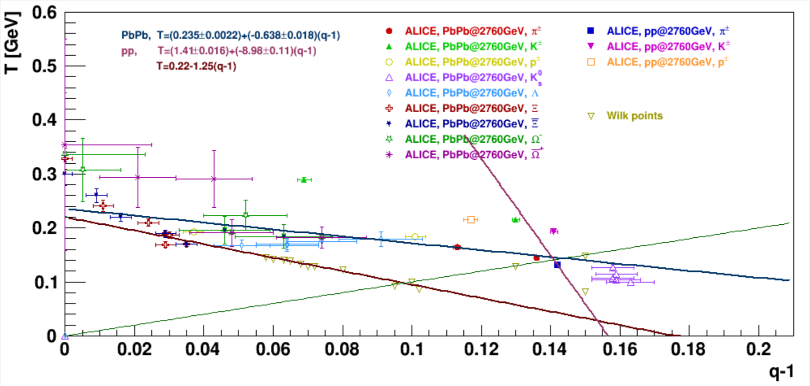
$$T = E\left(\sigma^2 - (q-1)\right)$$

pp: assume fix parameter NBD, $(q-1) = 1/(k+1)$, $\langle n \rangle = f(k+1)$, $T = E/\langle n \rangle$, and obtain

$$T = \frac{E}{f}(q-1)$$

Trends in Tsallis fits to data

G. Biro's collection, master thesis, 2015



Outline

Definition

Consider a coherent state defined by

$$| \, z \, \rangle = \sum_{n=0}^{\infty} \sqrt{p_n(t)} \, e^{in\Theta} \; | \, n \, \rangle \qquad (7)$$

with $z = \sqrt{t} e^{i\Theta}$. *("nonlinear coherent state")*

It overlaps with the n-quantum state:

$$|\langle \, n | z \, \rangle|^2 \; = \; p_n(t) \geq 0. \qquad (8)$$

Normalization

It is normalized,

$$\langle z|z\rangle = \sum_{n,m} \langle m|\sqrt{p_m p_n}\, e^{i(n-m)\Theta}\,|n\rangle = \sum_n p_n(t) = 1. \qquad (9)$$

The expectation value of a function of the number operator is

$$\langle z\,|\,\varphi(N)\,|\,z\rangle = \sum_n \varphi(n)\, p_n(t). \qquad (10)$$

This ensures that $p_n(t)$ is a probability distribution in n!

Completeness

We construct a complete set:

$$\int \frac{d^2z}{\pi}\,|\,z\,\rangle\,\langle\,z\,| \;=\; \int_0^\infty dt \int_0^{2\pi} \frac{d\Theta}{2\pi} \sum_{n,m} \sqrt{p_n p_m}\, e^{i(n-m)\Theta}\,|\,m\,\rangle\,\langle\,n\,|$$

$$= \int_0^\infty dt \sum_n p_n(t)\,|\,n\,\rangle\,\langle\,n\,| \;=\; \sum_n |\,n\,\rangle\,\langle\,n\,| \;=\; 1. \quad (11)$$

It is satisfied only if $\int_0^\infty dt\, p_n(t) \;=\; 1.$

This makes $p_n(t)$ to a probability distribution function of t!

CS as eigenstate to some operator

Eigenstate with eigenvalue z to

$$F \,|\, z \,\rangle \;=\; ag(\hat{n}) \,|\, z \,\rangle \;=\; z \,|\, z \,\rangle \,. \tag{12}$$

Here a is an annihilating ($a^\dagger$ is a creating) operator, and $\hat{n} = a^\dagger a$ is the number operator.
Its action on the CS:

$$F \,|\, z \,\rangle \;=\; \sum_{n=1}^{\infty} g(n) \sqrt{n p_n} \, e^{in\Theta} \,|\, n-1 \,\rangle \,, \tag{13}$$

can be re-indexed to

$$F \,|\, z \,\rangle \;=\; \sum_{n=0}^{\infty} g(n+1) \sqrt{(n+1) p_{n+1}} \, e^{i(n+1)\Theta} \,|\, n \,\rangle \,, \tag{14}$$

Recursion law

Compare this with

$$z \,|\, z \,\rangle \;=\; \sqrt{t}\, e^{i\Theta} \sum_{n=0}^{\infty} \sqrt{p_n}\, e^{in\Theta} \,|\, n \,\rangle \,, \tag{15}$$

to conclude that

$$p_n(t) \;=\; \frac{t}{ng(n)^2}\, p_{n-1}(t). \tag{16}$$

n-probability

Symmetric generalized binomial distributions, J.Math.Phys. 54 (2013) 123301

The recursion is solved by

$$p_n(t) \;=\; p_0(t)\,\frac{t^n}{n!}\prod_{j=1}^{n} g(j)^{-2}. \tag{17}$$

Here $p_0(t)$ can be obtained from the normalization condition.

Also the completeness constraint, $\int_0^\infty dt\, p_n(t) \;=\; 1$, has to be checked.

Outline

Traditional CS

The most known CS is defined by $\boldsymbol{g(n) = 1}$.

This results in a **Poisson** in n and **Euler-Gamma** in t:

$$p_n(t) \; = \; \frac{t^n}{n!} \, e^{-t}, \tag{18}$$

and $|\, z \,\rangle$ is an eigenstate to the $F = a$ annihilator.

NB coherent states

The negative binomial distribution (NBD),

$$p_n(t) = \binom{n+k}{n} (t/k)^n (1+t/k)^{-n-k-1}, \qquad (19)$$

is well normalized in n and, as an Euler-Beta distribution, also in t. From the recursion one obtains

$$g(n)^2 = \frac{t}{n} \frac{p_{n-1}}{p_n} = \frac{k+t}{k+n}, \qquad (20)$$

so this state satisfies

$$a\sqrt{\frac{k+|z|^2}{k+a^\dagger a}} \,|\,z\,\rangle = z \,|\,z\,\rangle. \qquad (21)$$

7 sources of NBD

1. Phase space cell statistics (Biró)
2. Squeeze parameter (Varró, Jackiw)
3. Wave packet statistics (Pratt, Csörgő, Zimányi)
4. Temperature superstatistics (Beck, Wilk)
5. KNO + pQCD (Dokshitzer, Dremin, Hegyi, Carruthers)
6. Tsallis/Rényi entropy canonical state (Rényi, Tsallis, ...)
7. Glittering Glasma (Gelis, Lappi, McLerran)

1. Phase Space Cell Statistics

Probability to find n particles in k cells, if altogether we have thrown N particles into K cells:

Pólya distribution

$$P_n = \frac{\binom{k+n}{n}\binom{K-k+N-n}{N-n}}{\binom{K+N+1}{N}}.$$

Necessary limit: $K \to \infty, N \to \infty$ while $f = N/K$ kept finite.

Prediction: $k+1 = \langle n \rangle / f \propto N_{\text{part}}$, if f is universal.

Here k is the number of observed phase space cells: from which the detected n particles seem to come.

2. Negative Binomial State (NBS)

Neg.Binom. state as a nonlinear coherent state:

$$|\, z, k \,\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k)}\, e^{in\Theta} \,|\, n \,\rangle \tag{22}$$

with

$$p_n(k) \;=\; \binom{k+n}{n} f^n \,(1+f)^{-n-k-1}. \tag{23}$$

NBS annihilated

$$a\,|\, z, k \,\rangle \;=\; \sqrt{f(k+1)}\, e^{i\Theta}\,|\, z, k+1 \,\rangle\,. \tag{24}$$

3. Wave packet statistics

HBT with onefold filled bosonic states gives a correlation factor of 2 at zero relative momentum. With M-fold occupation of the same state it reduces to

$$C_2(0) = 1 + \frac{1}{M} = 1 + \frac{1}{k+1}. \tag{25}$$

The logarithmic cumulants for an NBS, defined by $G(z) = \sum p_n z^n$ and $\ln G(z) = \sum_{n=1}^{\infty} C_n(z^n - 1)$, are

$$C_n = \frac{k+1}{n} \left(\frac{f}{1+f} \right)^n . \tag{26}$$

This is a $(k+1)$-fold overload of the simple Bose case, given if $k = 0$.

4. Superstatistics

Beck, Wilk

Thermodynamical β-fluctuation and n-fluctuations are related by Poisson transform:

$$\int_0^\infty \gamma(\beta) e^{-\beta\omega} d\beta \;=\; \sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n P_n(E). \tag{27}$$

In this way $\Delta\beta^2 / \langle\, \beta \,\rangle^2 = 1/(k+1)$.

5. pQCD

Dokshitzer, Dremin, Hegyi, Carruthers

KNO scaling + GLAP give nearly NBD with constant k, related to Λ_{QCD} and expressed by n-variance.

NBD is in fact slightly violated.

6. Canonical

Accepting Tsallis' or Rényi entropy as a formula, the usual canonical constraint on the average energy leads to

$$w(\omega) = \left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} \tag{28}$$

Using further assumptions about reservoir fluctuations, further entropy formulas can be constructed, as expectation values of formal logarithms, behaving additively (ARC).

7. Glittering Glasma: k-fold ropes

Gelis, Lappi, McLerran

$k = \kappa(N_c^2 - 1)Q_s^2R^2/2$ is about the number of tubes, makes NBS with this parameter.

NB coherent states

The negative binomial distribution (NBD),

$$p_n(t) \;=\; \binom{n+k}{n} (t/k)^n \, (1+t/k)^{-n-k-1}, \tag{29}$$

is well normalized in n and, as an Euler-Beta distribution, also in t. From the recursion one obtains

$$g(n)^2 \;=\; \frac{t}{n}\,\frac{p_{n-1}}{p_n} \;=\; \frac{k+t}{k+n}, \tag{30}$$

so this state satisfies

$$a\sqrt{\frac{k+|z|^2}{k+a^\dagger a}}\,|\,z\,\rangle \;=\; z\,|\,z\,\rangle\,. \tag{31}$$

su(1,1) structure in NBD

Rapidity-like notation: $t/k = \sinh^2 \zeta$;

$$p_n(t) = \binom{k+n}{n} \sinh^{2n} \zeta \cosh^{-2n-2k-2} \zeta.$$

$$|\, z \,\rangle = \cosh^{-k-1}(\zeta) \sum_{n=0}^{\infty} \sqrt{\binom{k+n}{n}} \left(\tanh(\zeta) e^{i\Theta}\right)^n |\, n \,\rangle . \qquad (32)$$

Using velocity $v = \tanh(\zeta)$ the overlap between two NBD Coh.States:

$$|\langle\, z_1 | z_2 \,\rangle|^2 = \left[1 + \gamma_1^2 \gamma_2^2 \left| v_1 e^{i\Theta_1} - v_2 e^{i\Theta_2} \right|^2 \right]^{-k-1} . \qquad (33)$$

2+1 dim relative velocity vector separates

2+1 dim vector representation

$$\vec{K} = \gamma_1\gamma_2\left(\vec{v}_1 - \vec{v}_2\right) = \frac{1}{\sqrt{k}}\left(\gamma_2 z_1 - \gamma_1 z_2\right). \tag{34}$$

Overlap written this way

$$|\langle z_1|z_2\rangle|^2 = \left[1 + \frac{1}{k}\left|\gamma_2 z_1 - \gamma_1 z_2\right|^2\right]^{-k-1} \to e^{-|z_1-z_2|^2}. \tag{35}$$

with $\gamma_i = \sqrt{1+|z_i|^2/k}$.
Particle properties of the vector $\vec{K}$:

$$\vec{K} = \frac{1}{m_1 m_2}\left(E_2\vec{P}_1 - E_1\vec{P}_2\right) \tag{36}$$

Its parallel component does not Lorentz transform: $\vec{v}\vec{K}' = \vec{v}\vec{K}$.

Negative Binomial States

in our own notation

$$|\, z_k, k \,\rangle \;=\; \sum_{n=0}^{\infty} \sqrt{p_n(k)} e^{in\Theta} \,|\, n \,\rangle \tag{37}$$

with $z_k = \sqrt{kf}\, e^{i\Theta}$ and

$$p_n(k) \;=\; \binom{k+n}{n} f^n \,(1+f)^{-n-k-1} \tag{38}$$

provides an average photon number $\langle\, n \,\rangle = f(k+1)$.

- excited geometric state: n over k
- intermediate number state: $f/(1+f)$
- eigenstate with complex eigenvalue

The Effect of Ladder Operator

Effect of annihilating a quantum:

$$a \,|\, z_k, k \,\rangle \;=\; \sum_{n=1}^{\infty} \sqrt{p_n} e^{in\Theta} \sqrt{n} \,|\, n-1 \,\rangle \;=\; \sum_{n=0}^{\infty} \sqrt{(n+1)p_{n+1}}\, e^{i\Theta}\, e^{in\Theta} \,|\, n \,\rangle \,. \tag{39}$$

Consider now

$$(n+1)\binom{k+n+1}{n+1} f^{n+1}(1+f)^{-(n+1)-k-1} = f(k+1)\binom{k+1+n}{k+1} f^{n}(1+f)^{-n-(k+1)-1}. \tag{40}$$

Here we recognize $z_{k+1} = \sqrt{f(k+1)}e^{i\Theta}$ as a factor and arrive at

NBS annihilated

$$a \,|\, z_k, k \,\rangle \;=\; z_{k+1} \,|\, z_{k+1}, k+1 \,\rangle \,. \tag{41}$$

NBS as eigenstate of what?

The action of another operator

$$\sqrt{\hat{N}+k+1}\,|\,z,k\,\rangle \;=\; \sqrt{(k+1)(1+f)}\,|\,z,k+1\,\rangle\,, \tag{42}$$

based on the relation

$$(k+1+n)\binom{k+n}{k} \;=\; (k+1)\binom{k+1+n}{k+1}. \tag{43}$$

This helps to recognize:

NBS eigenvalue equation

$$\left(\sqrt{f}\,(\hat{n}+k+1) - \sqrt{1+f}\,e^{-i\Theta}\,\sqrt{\hat{n}+k+1}\,a\right)|\,z,k\,\rangle \;=\; 0. \tag{44}$$

Our NBS algebra

In the previous equation the following operator occurs:

$$K_- = \sqrt{\hat{n}+k+1}\, a, \qquad K_+ = K_-^\dagger = a^\dagger \sqrt{\hat{n}+k+1}.$$

The commutator,

$$[K_-, K_+] = (\hat{n}+1)(\hat{n}+k+1) - \hat{n}(\hat{n}+k) = 2\hat{N}+k+1 = 2K_0 \quad (45)$$

defines $K_0 = \hat{n} + (k+1)/2$.

With $\alpha = \sqrt{(1+f)/f}\, e^{-i\Theta}$ we get

OUR eigenvalue equation

$$(\alpha K_- - K_0) \,|\, z, k \,\rangle = \frac{k+1}{2} \cdot |\, z, k \,\rangle \,. \quad (46)$$

SU(1,1) algebra for NBS

The commutators form an SU(1,1) algebra:

Commutators

$$
\begin{aligned}
[K_0, K_+] &= K_+ \\
[K_0, K_-] &= -K_- \\
[K_-, K_+] &= 2K_0
\end{aligned}
\tag{47}
$$

The Casimir operator is given as: $Q = K_0^2 - K_0 - K_+ K_-$.

$|\, NBS \,\rangle$ created from $|\, 0 \,\rangle$: 1. preliminaries

Operator identity

$$e^{A+B} \;=\; e^{-\lambda/2}\, e^{A}\, e^{B} \qquad \text{if} \qquad [A,B] \;=\; \lambda \quad (\text{const.}) \tag{48}$$

We choose $A = \alpha\, z g(\hat{n})\, a^{\dagger}$ and $B = -\beta\, z^{*} a\, \frac{1}{g(\hat{n})} \neq -A^{\dagger}$.

Commutator:

$$\lambda = [A,B] = -|z|^2 \alpha\beta\, g(\hat{n}) a^{\dagger} a \frac{1}{g(\hat{n})} + |z|^2 \alpha\beta\, a \frac{1}{g(\hat{n})} g(\hat{n}) a^{\dagger} = |z|^2 \alpha\beta. \tag{49}$$

$|\, NBS \,\rangle$ created from $|\, 0 \,\rangle$: 2. evolution operator

$$U \;=\; e^{\Phi/2+A+B} \;=\; e^{(\Phi-\lambda)/2}\, e^A\, e^B. \tag{50}$$

Here $e^B \,|\, 0 \,\rangle = |\, 0 \,\rangle$ due to $B \,|\, 0 \,\rangle = 0$.

$$U \,|\, 0 \,\rangle \;=\; e^{(\Phi-\alpha\beta\,|z|^2)/2} \sum_{n=0}^{\infty} \frac{\alpha^n z^n}{n!} \left(g(\hat{n}) a^\dagger \right)^n \,|\, 0 \,\rangle\,. \tag{51}$$

$|\, NBS \,\rangle$ created from $|\, 0 \,\rangle$: 3. n-photon state

We have

$$\left(g(\hat{n})a^{\dagger}\right)^{n} |\, 0 \,\rangle \;=\; g(n) \cdot \ldots \cdot g(1)\, \sqrt{n!}\; |\, n \,\rangle \,. \tag{52}$$

Regarding the form

$$U |\, 0 \,\rangle \;=\; \sum_{n=0}^{\infty} \sqrt{u_n}\, e^{in\Theta}\, |\, n \,\rangle \,, \tag{53}$$

with $z = \sqrt{t}\, e^{i\Theta}$, and using $\boldsymbol{g(\hat{n}) = \xi \sqrt{\hat{n} + k}}$ we obtain

$$u_n = e^{\Phi - \alpha\beta\, t} \left(\alpha^2 \xi^2 t\right)^{n} \binom{k+n}{n}. \tag{54}$$

$|\, NBS \,\rangle$ created from $|\, 0 \,\rangle$: 4. normalization

For $||U\,|\,0\,\rangle\,||^2 = 1$ one needs

$$\sum_{n=0}^{\infty} u_n \;=\; e^{\Phi-\alpha\beta t}\,\left(1-\alpha^2\xi^2 t\right)^{-(k+1)} \;=\; 1. \tag{55}$$

From this we express

$$\alpha^2\xi^2 t = 1 - e^{\frac{1}{k+1}(\Phi-\alpha\beta t)} \;=\; 1-w, \tag{56}$$

and gain

negative binomial distribution

$$u_n \;=\; \binom{n+k}{n}\, w^{k+1}\,(1-w)^n. \tag{57}$$

$|\, NBS \,\rangle$ created from $|\, 0 \,\rangle$: 5. superstatistics

For the superstatistics normalization, we utilize

$$\int_0^\infty dt\, u_n(t) = \binom{k+n}{n} \int_0^\infty dt\, w^{k+1}(1-w)^n = \binom{k+n}{n} k \int_0^1 dw\, w^{k-1}(1-w)^n = 1. \tag{58}$$

This is achieved if

$$dt = -k\frac{dw}{w^2}, \qquad \longrightarrow \qquad t = k\left(\frac{1}{w} - 1\right), \tag{59}$$

or expressing $w(t)$ if

$$w = \frac{1}{1+t/k} = \frac{k}{t+k}. \tag{60}$$

$|\, NBS \,\rangle$ created from $|\, 0 \,\rangle$: 6. conclusion

In this way α, β and ξ remain undetermined. A purposeful choice is $\alpha = \beta = 1/\sqrt{t} = 1/|z|$ with $\xi = 1/\sqrt{t+k}$.

The log of the evolution operator becomes in this case

neither hermitic nor anti-hermitic

$$\ln U \;=\; -\frac{k+1}{2} \ln\left(1 + t/k\right) + \frac{1}{2} + e^{i\Theta} \sqrt{\frac{\hat{n}+k}{t+k}}\, a^{\dagger} \;-\; e^{-i\Theta}\, a \sqrt{\frac{t+k}{\hat{n}+k}}. \tag{62}$$

U may be connected to a Hamiltonian...

$|\,NBS\,\rangle$ created from $|\,0\,\rangle$:

7. physics behind

The anti-hermitic part gives a guess for the Hamiltonian

$$\frac{i}{\hbar}\int H d\tau = \frac{1}{2}\left(\ln U - (\ln U)^{\dagger}\right) = e^{i\Theta}\left(f(\hat{n}) + \frac{1}{f(\hat{n})}\right)a^{\dagger} - e^{-i\Theta}\,a\left(f(\hat{n}) + \frac{1}{f(\hat{n})}\right). \tag{63}$$

The hermitic part means non-conserved particle number...

Summary

- A class of coherent states designed for a given n pdf is also a superstatistics in $t = |z|^2$;
- The NB coherent state is an eigenstate for the regularized phase operator.
- An su(1,1) structure is inherent in the NB coherent state.
- Hamiltonians combined from K_-, K_0 and K_+ operators are likely to produce NB distributed bosons.